Abrashkin's work on the higher ramification filtration
Part 2: The filtration

Kevin Keating
Department of Mathematics
University of Florida

May 31, 2024

Our story so far . . .

K is a local field of characteristic $p > 2$. Hence $K \cong k((t))$, with $k \cong \mathbb{F}_{p^d}$ for some $d \geq 1$.

Let $K[p]$ be the maximal Galois extension of K whose Galois group is a pro-p group.

Then $\mathrm{Gal}(K[p]/K)$ is a free pro-p group.

Let $K(p)/K$ be the largest Galois subextension of $K[p]/K$ such that

- $\mathrm{Gal}(K(p)/K)$ has nilpotence class 2.
- $\mathrm{Gal}(K(p)/K)$ has exponent p.

The goal is to give a description of the ramification filtration of $\mathrm{Gal}(K(p)/K)$.

Generators for Gal($K[p]/K$)

Let $K(p)^{ab}/K$ be the maximal abelian subextension of $K(p)/K$; then $K(p)^{ab}/K$ is the maximal elementary abelian subextension of $K[p]/K$.

By class field theory we have

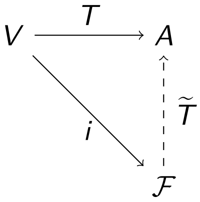
$$\mathrm{Gal}(K(p)^{ab}/K) \cong K^\times/(K^\times)^p.$$

Thus Gal($K[p]/K$) is free pro-p on the module $K^\times/(K^\times)^p$. In other words, the free pro-p group Gal($K[p]/K$) is determined up to isomorphism by the condition that its maximal elementary abelian quotient is $K^\times/(K^\times)^p$.

Free Lie algebras

Definition

Let V be a vector space over $\mathbb{F}_p$. A free Lie algebra on V consists of a Lie algebra $\mathcal{F}$ over $\mathbb{F}_p$ and a one-to-one linear map $i : V \to \mathcal{F}$ which satisfy the following universal property: Let A be a Lie algebra over $\mathbb{F}_p$ and let $T : V \to A$ be a linear map. Then there is a unique Lie algebra homomorphism $\widetilde{T} : \mathcal{F} \to A$ such that $T = \widetilde{T} \circ i$.



A quotient of a free Lie algebra

Let $\widetilde{\mathcal{L}}$ be a free $\mathbb{F}_p$-Lie algebra on $K^\times/(K^\times)^p$. Let $\mathcal{I}$ be the ideal in $\mathcal{L}$ generated by 3rd commutators, and set $\mathcal{L} = \widetilde{\mathcal{L}}/\mathcal{I}$.

Then $\mathcal{L}$ is a Lie algebra over $\mathbb{F}_p$ which is nilpotent of class 2. Hence the Baker-Campbell-Hausdorff formula defines a group operation $*$ on $\mathcal{L}$, which is given by

$$x * y = x + y + \frac{1}{2} \cdot [x, y].$$

We get a filtered pro-unipotent group $\mathcal{G}$ over $\mathbb{F}_p$ such that $(\mathcal{L}, *) = \mathcal{G}(\mathbb{F}_p)$.

Local class field theory induces an isomorphism between $\mathcal{L}/[\mathcal{L}, \mathcal{L}] \cong K^\times/(K^\times)^p$ and $\mathrm{Gal}(K(p)^{ab}/K)$. It follows from the universal properties of $\mathrm{Gal}(K(p)/K)$ and $\mathcal{L}$ that

$$(\mathcal{L}, *) \cong \mathrm{Gal}(K(p)/K).$$

Abrashkin gives a filtration on $\mathcal{L}$ which maps to the upper ramification filtration of $\mathrm{Gal}(K(p)/K)$ under this isomorphism.

Schmid's isomorphism

Recall that $K = k((t))$. For $\kappa \in K^\times$ define $\psi_\kappa : K \to \mathbb{F}_p$ by

$$\psi_\kappa(c) = \mathrm{Tr}_{k/\mathbb{F}_p}\left(\mathrm{Res}_0\left(c\frac{d\kappa}{\kappa}\right)\right),$$

where

$$\frac{d\kappa}{\kappa} = d\log\kappa = \frac{\kappa'(t)}{\kappa(t)}\,dt$$

and Res_0 denotes the residue of the differential form at $t = 0$.

Then ψ_κ is a homomorphism such that $\wp(K) \subset \ker\psi_\kappa$, where $\wp(x) = x^p - x$ is the Artin-Schreier operator.

Furthermore, if $\lambda \in K^\times$ then $\psi_{\kappa\lambda^p} = \psi_\kappa$.

The map $\kappa \mapsto \psi_\kappa$ induces an isomorphism

$$K^\times/(K^\times)^p \cong \mathrm{Hom}_{\mathbb{F}_p}(K/\wp(K), \mathbb{F}_p).$$

An explicit description of $(K^\times/(K^\times)^p) \otimes_{\mathbb{F}_p} k$

Set $\mathbb{Z}^+(p) = \{a \in \mathbb{Z} : a \geq 1,\ p \nmid a\}$ and let $\alpha_0 \in k$ with $\mathrm{Tr}_{k/\mathbb{F}_p}(\alpha_0) = 1$. Then

$$K/\wp(K) \cong \left(\bigoplus_{a \in \mathbb{Z}^+(p)} kt^{-a} \right) \oplus \mathbb{F}_p \alpha_0.$$

(In fact the right side is a complement of $\wp(K)$ in $(K, +)$.)

It follows using Schmid's isomorphism that

$$\begin{aligned}
(K^\times/(K^\times)^p) \otimes_{\mathbb{F}_p} k &\cong \mathrm{Hom}_{\mathbb{F}_p}(K/\wp(K), \mathbb{F}_p) \otimes_{\mathbb{F}_p} k \\
&\cong \mathrm{Hom}_{\mathbb{F}_p}(K/\wp(K), k) \\
&\cong \left(\prod_{a \in \mathbb{Z}^+(p)} \mathrm{Hom}_{\mathbb{F}_p}(kt^{-a}, k) \right) \times \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p \alpha_0, k).
\end{aligned}$$

Generators for $(K^\times/(K^\times)^p) \otimes_{\mathbb{F}_p} k$

Recall that $k \cong \mathbb{F}_{p^d}$. For $a \in \mathbb{Z}^+(p)$ and $n \in \mathbb{Z}/d\mathbb{Z}$ define $D_{a,n} \in \mathrm{Hom}_{\mathbb{F}_p}(kt^{-a}, k)$ by $D_{a,n}(rt^{-a}) = r^{p^n}$. Then $D_{a,0}, \dots, D_{a,d-1}$ is a k-basis for $\mathrm{Hom}_{\mathbb{F}_p}(kt^{-a}, k)$.

In addition, define $D_{0,n} : \mathbb{F}_p\alpha_0 \to k$ by $D_{0,0}(r\alpha_0) = r^{p^n}$. Then each $D_{0,n}$ generates the k-vector space $\mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p\alpha_0, k)$.

Extend $D_{a,n}$ to $\mathrm{Hom}_{\mathbb{F}_p}(K/\wp(K), k)$ by setting $D_{a,n}(rt^{-b}) = 0$ for $r \in k$ and $b \neq a$. Then

$$S = \{D_{a,n} : a \in \mathbb{Z}^+(p),\ n \in \mathbb{Z}/d\mathbb{Z}\} \cup \{D_{0,0}\}$$

is a topological basis for

$$\mathrm{Hom}_{\mathbb{F}_p}(K/\wp(K), k) \cong (K^\times/(K^\times)^p) \otimes_{\mathbb{F}_p} k.$$

Therefore S is a topological generating set for the k-Lie algebra $\mathcal{L}_k = \mathcal{L} \otimes_{\mathbb{F}_p} k$.

It follows that $\mathcal{L}_k$ is generated as a topological vector space over k by $S \cup [S, S]$.

Generators for $K(p)/K$

Let $\mathcal{G}$ be the pro-algebraic group over $\mathbb{F}_p$ given by the operation $*$ on $\mathcal{L}$. Then

$$\mathcal{G}(\mathbb{F}_p) \cong (\mathcal{L}, *) \cong \mathrm{Gal}(K(p)/K).$$

Set $\mathbb{Z}_0^+(p) = \mathbb{Z}^+(p) \cup \{0\}$ and define $e \in \mathcal{G}(K) = \mathcal{L}\widehat{\otimes}_{\mathbb{F}_p} K$ by

$$e = \sum_{a \in \mathbb{Z}_0^+(p)} t^{-a} D_{a,0}.$$

Let $x \in \mathcal{G}(K^{sep}) = \mathcal{L}\widehat{\otimes}_{\mathbb{F}_p} K^{sep}$ satisfy $\phi(x) = e * x.$ The coordinates of x generate a Galois extension E/K, and there is a homomorphism $\theta_x : \mathrm{Gal}(K(p)/K) \to (\mathcal{L}, *)$ which factors through $\mathrm{Gal}(E/K)$.

Recall that $K(p)^{ab}/K$ is the maximal elementary abelian p-extension of K. By the choice of e, θ_x induces an isomorphism from $\mathrm{Gal}(K(p)^{ab}/K)$ onto $\mathcal{L}/[\mathcal{L}, \mathcal{L}]$.

It follows from the universal properties of $\mathrm{Gal}(K(p)/K)$ and $\mathcal{L}$ that θ_x is an isomorphism, and $E = K(p).$ Therefore

$$\mathrm{Gal}(K(p)/K) \cong (\mathcal{L}, *).$$

Generators for $K(p)/K$ ...

We have $K \subset K(p)^{ab} \subset K(p)$, with $K(p)^{ab}/K$ and $K(p)/K(p)^{ab}$ elementary abelian p-extensions.

Hence $K(p)^{ab}/K$ and $K(p)/K(p)^{ab}$ can be described in terms of Artin-Schreier theory.

Example

Suppose $k = \mathbb{F}_p$. For $a \in \mathbb{Z}_0^+(p)$ let $x_a \in K^{sep}$ be a root of the Artin-Schreier polynomial $X^p - X - t^{-a}$. Then $K(p)^{ab}$ is generated over K by $\{x_a : a \in \mathbb{Z}_0^+(p)\}$.

For $a, b \in \mathbb{Z}_0^+(p)$ with $a < b$ let $y_{a,b}$ be a root of

$$X^p - X - \tfrac{1}{2}(t^{-a}x_b - t^{-b}x_a).$$

Then $K(p)$ is generated over $K(p)^{ab}$ by

$$\{y_{a,b} : a, b \in \mathbb{Z}_0^+(p),\ a < b\}.$$

Heisenberg subextensions of $K(p)/K$

Assume once again that $k = \mathbb{F}_p$. Let $a, b \in \mathbb{Z}_0^+(p)$ with $a < b$ and set $E_{a,b} = K(x_a, x_b, y_{a,b})$. Then $E_{a,b}/K$ is a Galois extension whose Galois group is the Heisenberg group of order p^3. Hence $K(p)$ is a compositum of Heisenberg extensions of K.

Let $\mathcal{I}_{a,b}$ be the ideal in $\mathcal{L}$ generated by the set

$$\{D_{c,0} : c \in \mathbb{Z}_0^+(p),\ c \neq a,\ c \neq b\}.$$

Set $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{I}_{a,b}$ and $G = \mathrm{Gal}(E_{a,b}/K)$. Then $G \cong (\overline{\mathcal{L}}, *)$.

The only generators of $\mathrm{Gal}(K(p)/K) \cong (\mathcal{L}, *)$ which act nontrivially on $E_{a,b}$ are $D_{a,0}$ and $D_{b,0}$. Denote their images in G by $\overline{D}_{a,0}, \overline{D}_{b,0}$. Then $\overline{\mathcal{L}}$ is the $\mathbb{F}_p$-span of $\overline{D}_{a,0}, \overline{D}_{b,0}, [\overline{D}_{a,0}, \overline{D}_{b,0}]$.

Generators for ramification ideals

Let $\alpha \in \mathbb{Q}$ with $\alpha > 0$ and let $N \geq 0$.

If $\alpha \notin \mathbb{Z}$ set

$$\mathcal{F}_{\alpha,N} = \sum_{\substack{\alpha = a_1 + a_2 p^{-m} \\ 0 \leq m \leq N}} \frac{a_1}{2} \cdot [D_{a_1,0}, D_{a_2,m}],$$

while if $\alpha \in \mathbb{Z}$ set

$$\mathcal{F}_{\alpha,N} = \alpha \cdot D_{\alpha,0} + \sum_{\substack{\alpha = a_1 + a_2 p^{-m} \\ 0 \leq m \leq N}} \frac{a_1}{2} \cdot [D_{a_1,0}, D_{a_2,m}].$$

In both sums we require $a_1, a_2 \in \mathbb{Z}_0^+(p)$.

Since $D_{a,n} \in \mathcal{L}_k = \mathcal{L} \otimes_{\mathbb{F}_p} k$ we have $\mathcal{F}_{\alpha,N} \in \mathcal{L}_k$.

Note that if $p^N \alpha \notin \mathbb{Z}$ then $\mathcal{F}_{\alpha,N} = 0$.

Some examples

Example

Let $\alpha = 2 + p^{-1}$ and $N \geq 1$. Then

$$\alpha = 2 + 1 \cdot p^{-1} = 1 + (p+1)p^{-1} = 0 + (2p+1)p^{-1}$$

$$\begin{aligned}\mathcal{F}_{\alpha,N} &= \tfrac{2}{2} \cdot [D_{2,0}, D_{1,1}] + \tfrac{1}{2} \cdot [D_{1,0}, D_{p+1,1}] + \tfrac{0}{2} \cdot [D_{0,0}, D_{2p+1,1}] \\ &= [D_{2,0}, D_{1,1}] + \tfrac{1}{2} \cdot [D_{1,0}, D_{p+1,1}].\end{aligned}$$

Example

Let $\alpha = 2$ and $N = 1$. Then

$$\alpha = 0 + 2 \cdot p^{-0} = 1 + 1 \cdot p^{-0} = 2 + 0 \cdot p^{-0} = 2 + 0 \cdot p^{-1}$$

$$\begin{aligned}\mathcal{F}_{\alpha,N} &= 2 \cdot D_{2,0} + \tfrac{0}{2} \cdot [D_{0,0}, D_{2,0}] + \tfrac{1}{2} \cdot [D_{1,0}, D_{1,0}] + \tfrac{2}{2} \cdot [D_{2,0}, D_{0,0}] \\ &\qquad + \tfrac{2}{2} \cdot [D_{2,0}, D_{0,1}] \\ &= 2 \cdot D_{2,0} + [D_{2,0}, D_{0,0}] + [D_{2,0}, D_{0,1}].\end{aligned}$$

The ramification filtration

For $v \in \mathbb{Q}^+$ let $\mathcal{L}_k[v]$ be the Lie ideal in $\mathcal{L}_k$ generated by

$$\{\mathcal{F}_{\alpha,N} : \alpha \geq v,\ N \geq 0\}.$$

Let $\mathcal{L}(v)$ be the smallest $\mathbb{F}_p$-subspace of $\mathcal{L}$ such that $\mathcal{L}_k[v] \subset \mathcal{L}(v) \otimes_{\mathbb{F}_p} k$. Then $\mathcal{L}(v)$ is an ideal in $\mathcal{L}$.

Let $\mathcal{L}^v$ be the ideal in $\mathcal{L}$ that corresponds to the vth upper ramification subgroup of $\mathrm{Gal}(K(p)/K)$ under the isomorphism $\mathrm{Gal}(K(p)/K) \cong (\mathcal{L}, *)$.

Theorem (Abrashkin)

For every $v \in \mathbb{Q}^+$ we have $\mathcal{L}^v = \mathcal{L}(v)$.

A Heisenberg extension

Let $p > 2$ and set $K = \mathbb{F}_p((t))$; then $k = \mathbb{F}_p$. Let $0 < a < b$ with $p \nmid a$, $p \nmid b$.

Consider the Heisenberg extension $E_{a,b}/K$ that we constructed earlier: $E_{a,b} = K(x_a, x_b, y_{a,b})$, where

$$x_a^p - x_a = t^{-a}$$
$$x_b^p - x_b = t^{-b}$$
$$y_{a,b}^p - y_{a,b} = \tfrac{1}{2}(t^{-a}x_b - t^{-b}x_a).$$

Then $E_{a,b}/K$ is the subextension of $K(p)/K$ which corresponds to the ideal $\mathcal{I}_{a,b} \subset \mathcal{L}$. Set $G = \mathrm{Gal}(E_{a,b}/K)$. Then $G \cong (\overline{\mathcal{L}}, *)$, where $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{I}_{a,b}$.

The only generators of $\mathrm{Gal}(K(p)/K) \cong (\mathcal{L}, *)$ which act nontrivially on $E_{a,b}$ are $D_{a,0}$ and $D_{b,0}$. Denote their images in G by $\overline{D}_{a,0}, \overline{D}_{b,0}$.

For $\alpha \in \mathbb{Q}^+$, $N \geq 0$ let $\overline{\mathcal{F}}_{\alpha,N}$ denote the image of $\mathcal{F}_{\alpha,N}$ in $\overline{\mathcal{L}}$. Then the only values of α for which $\overline{\mathcal{F}}_{\alpha,N} \neq 0$ are $a, b, a+b, a+bp^{-m}, b+ap^{-m}$ with $1 \leq m \leq N$.

A Heisenberg extension ...

For all $N \geq 0$ we get

$$\overline{\mathcal{F}}_{a,N} = a \cdot \overline{D}_{a,0}$$
$$\overline{\mathcal{F}}_{b,N} = b \cdot \overline{D}_{b,0}$$
$$\overline{\mathcal{F}}_{a+b,N} = \tfrac{a}{2} \cdot [\overline{D}_{a,0}, \overline{D}_{b,0}] + \tfrac{b}{2} \cdot [\overline{D}_{b,0}, \overline{D}_{a,0}] = \tfrac{1}{2}(a-b) \cdot [\overline{D}_{a,0}, \overline{D}_{b,0}].$$

For $N \geq m \geq 1$ we get

$$\overline{\mathcal{F}}_{a+bp^{-m},N} = \tfrac{a}{2} \cdot [\overline{D}_{a,0}, \overline{D}_{b,m}] = \tfrac{a}{2} \cdot [\overline{D}_{a,0}, \overline{D}_{b,0}]$$
$$\overline{\mathcal{F}}_{b+ap^{-m},N} = \tfrac{b}{2} \cdot [\overline{D}_{b,0}, \overline{D}_{a,m}]. = -\tfrac{b}{2} \cdot [\overline{D}_{a,0}, \overline{D}_{b,0}].$$

If $a \not\equiv b \pmod p$ then $\overline{\mathcal{L}}^{a+b}$ is the $\mathbb{F}_p$-span of $\overline{\mathcal{F}}_{a+b,0}$, and $\overline{\mathcal{L}}^{a+b+\epsilon} = \{0\}$.

If $a \equiv b \pmod p$ then $\overline{\mathcal{L}}^{b+ap^{-1}}$ is the $\mathbb{F}_p$-span of $\overline{\mathcal{F}}_{b+ap^{-1},1}$, and $\overline{\mathcal{L}}^{b+ap^{-1}+\epsilon} = \{0\}$.

In both cases, $\overline{\mathcal{L}}^{a}/\overline{\mathcal{L}}^{a+\epsilon}$, $\overline{\mathcal{L}}^{b}/\overline{\mathcal{L}}^{b+\epsilon}$ have order p and are spanned by $\overline{\mathcal{F}}_a$, $\overline{\mathcal{F}}_b$. It follows that $E_{a,b}/K$ has upper ramification breaks $a, b, a+b$ if $a \not\equiv b \pmod p$, and $a, b, b+ap^{-1}$ if $a \equiv b \pmod p$.

Another Heisenberg extension

Let $p > 2$ and set $K = \mathbb{F}_{p^2}((t))$; thus $k = \mathbb{F}_{p^2}$. Let $a > 0$ with $p \nmid a$ and let $c \in \mathbb{F}_{p^2}^{\times}$ satisfy $c^p = -c$.

Consider the Heisenberg extension E/K defined by $E = K(x_1, x_2, x_3)$, where $x_1^p - x_1 = t^{-a}$, $x_2^p - x_2 = ct^{-a}$, and $x_3^p - x_3 = \frac{1}{2}(t^{-a}x_2 - ct^{-a}x_1)$. There is an ideal $\mathcal{J} \subset \mathcal{L}$ which corresponds to the subextension E/K of $K(p)/K$. Setting $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{J}$ We get $\mathrm{Gal}(E/K) \cong (\overline{\mathcal{L}}, *)$.

Similar to the previous example, we find that $D_{a,0}, D_{a,1}$ are the only generators of $\mathrm{Gal}(K(p)/K) \cong (\mathcal{L}, *)$ whose images $\overline{D}_{a,0}, \overline{D}_{a,1}$ in $\overline{\mathcal{L}}_k = \overline{\mathcal{L}} \otimes_{\mathbb{F}_p} k$ are nontrivial.

Let $\overline{\mathcal{F}}_{\alpha,N}$ denote the image of $\mathcal{F}_{\alpha,N}$ in $\overline{\mathcal{L}}_k$. The only values of α for which $\overline{\mathcal{F}}_{\alpha,N} \neq 0$ are $a, a + ap^{-m}$ with $m \geq 1$ odd. We find that

$$\overline{\mathcal{F}}_{a,N} = a \cdot \overline{D}_{a,0} \qquad \text{for } N \geq 0$$
$$\overline{\mathcal{F}}_{a+ap^{-m},N} = \tfrac{a}{2} \cdot [\overline{D}_{a,0}, \overline{D}_{a,1}] \text{ for } 1 \leq m \leq N.$$

Hence $\overline{\mathcal{L}}_k[v] = \overline{\mathcal{L}}_k$ for $0 < v \leq a$, $\overline{\mathcal{L}}_k[v] = \mathbb{F}_{p^2} \cdot [\overline{D}_{a,0}, \overline{D}_{a,1}]$ for $a < v \leq a + ap^{-1}$, and $\overline{\mathcal{L}}_k[v] = \{0\}$ for $a + ap^{-1} < v$.

Another Heisenberg extension ...

Define homomorphisms $\phi : K/\wp(K) \to \mathbb{F}_p$, $\psi : K/\wp(K) \to \mathbb{F}_p$ by setting

$$\phi(rt^{-b}) = \begin{cases} 0 & (b \neq a) \\ \mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(r) & (b = a) \end{cases}$$

$$\psi(rt^{-b}) = \begin{cases} 0 & (b \neq a) \\ \mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(cr) & (b = a) \end{cases}$$

for $r \in \mathbb{F}_{p^2}$ and $b \in \mathbb{Z}_0^+(p)$.

Then $\phi, \psi \in \mathcal{L}$ induce elements $\overline{\phi}, \overline{\psi}$ of $\overline{\mathcal{L}}$ such that $\overline{D}_{a,0} = \frac{1}{2}\overline{\phi} + \frac{1}{2c}\overline{\psi}$ and $\overline{D}_{a,1} = \frac{1}{2}\overline{\phi} - \frac{1}{2c}\overline{\psi}$.

It follows that $\overline{\mathcal{L}}^v = \overline{\mathcal{L}}(v) = \overline{\mathcal{L}}$ for $0 < v \leq a$, $\overline{\mathcal{L}}^v = \overline{\mathcal{L}}(v) = \langle \overline{\phi}, \overline{\psi} \rangle$ for $a < v \leq a + ap^{-1}$, and $\overline{\mathcal{L}}^v = \overline{\mathcal{L}}(v) = \{0\}$ for $a + ap^{-1} < v$.

Thus a is an upper ramification break of E/K with multiplicity 2, and $a + ap^{-1}$ is an upper ramification break of E/K with multiplicity 1.

A bigger extension

Set $K = \mathbb{F}_5((t))$, so that $k = \mathbb{F}_5$. Consider the extension E/K defined by $E = K(x_1, x_2, x_3, x_4, x_5, x_6)$, where

$$\begin{aligned} x_1^5 - x_1 &= t^{-1} \\ x_2^5 - x_2 &= t^{-2} \\ x_3^5 - x_3 &= t^{-3} \\ x_4^5 - x_4 &= \tfrac{1}{2}(t^{-1}x_2 - t^{-2}x_1) \\ x_5^5 - x_5 &= \tfrac{1}{2}(t^{-1}x_3 - t^{-3}x_1) \\ x_6^5 - x_6 &= \tfrac{1}{2}(t^{-2}x_3 - t^{-3}x_2). \end{aligned}$$

Similar to the previous examples, we find that E/K is the subextension of $K(p)/K$ corresponding to an ideal $\mathcal{J} \subset \mathcal{L}$.

Then the Lie algebra $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{J}$ is generated by $\overline{D}_{1,0}, \overline{D}_{2,0}, \overline{D}_{3,0}$.

Let $\overline{\mathcal{F}}_{\alpha,N}$ denote the image of $\mathcal{F}_{\alpha,N}$ in G. The only values of $\alpha \in \mathbb{Q}^+$ with $\overline{\mathcal{F}}_{\alpha,N} \neq 0$ are

$1, 2, 3, 4, 5, 1+2\cdot 5^{-m}, 1+3\cdot 5^{-m}, 2+1\cdot 5^{-m}, 2+3\cdot 5^{-m}, 3+1\cdot 5^{-m}, 3+2\cdot 5^{-m}.$

A bigger extension ...

The corresponding ramification ideal generators are

$$
\begin{aligned}
\overline{\mathcal{F}}_{1,N} &= \overline{D}_{1,0} \\
\overline{\mathcal{F}}_{2,N} &= \overline{D}_{2,0} \\
\overline{\mathcal{F}}_{3,N} &= \overline{D}_{3,0} + \tfrac{1}{2}[\overline{D}_{1,0}, \overline{D}_{2,0}] + \tfrac{2}{2}[\overline{D}_{2,0}, \overline{D}_{1,0}] = \overline{D}_{3,0} - 2[\overline{D}_{1,0}, \overline{D}_{2,0}] \\
\overline{\mathcal{F}}_{4,N} &= \tfrac{1}{2}[\overline{D}_{1,0}, \overline{D}_{3,0}] + \tfrac{3}{2}[\overline{D}_{3,0}, \overline{D}_{1,0}] = -[\overline{D}_{1,0}, \overline{D}_{3,0}] \\
\overline{\mathcal{F}}_{5,N} &= \tfrac{2}{2}[\overline{D}_{2,0}, \overline{D}_{3,0}] + \tfrac{3}{2}[\overline{D}_{3,0}, \overline{D}_{2,0}] = 2[\overline{D}_{2,0}, \overline{D}_{3,0}]
\end{aligned}
$$

for all $N \geq 0$, and

$$
\begin{aligned}
\overline{\mathcal{F}}_{1+2\cdot 5^{-m},N} &= \tfrac{1}{2}[\overline{D}_{1,0}, \overline{D}_{2,0}] \\
\overline{\mathcal{F}}_{1+3\cdot 5^{-m},N} &= \tfrac{1}{2}[\overline{D}_{1,0}, \overline{D}_{3,0}] \\
\overline{\mathcal{F}}_{2+1\cdot 5^{-m},N} &= \tfrac{2}{2}[\overline{D}_{2,0}, \overline{D}_{1,0}] = -[\overline{D}_{1,0}, \overline{D}_{2,0}] \\
\overline{\mathcal{F}}_{2+3\cdot 5^{-m},N} &= \tfrac{2}{2}[\overline{D}_{2,0}, \overline{D}_{3,0}] = [\overline{D}_{2,0}, \overline{D}_{3,0}] \\
\overline{\mathcal{F}}_{3+1\cdot 5^{-m},N} &= \tfrac{3}{2}[\overline{D}_{3,0}, \overline{D}_{1,0}] = [\overline{D}_{1,0}, \overline{D}_{3,0}] \\
\overline{\mathcal{F}}_{3+2\cdot 5^{-m},N} &= \tfrac{3}{2}[\overline{D}_{3,0}, \overline{D}_{2,0}] = [\overline{D}_{2,0}, \overline{D}_{3,0}]
\end{aligned}
$$

for $1 \leq m \leq N$.

A bigger extension . . . . . .

Hence the ramification ideals are

$$\overline{\mathcal{L}}^v = \begin{cases} \overline{\mathcal{L}} & (v \leq 1) \\ \langle [\overline{D}_{2,0}, \overline{D}_{3,0}], [\overline{D}_{1,0}, \overline{D}_{3,0}], [\overline{D}_{1,0}, \overline{D}_{2,0}], \overline{D}_{3,0}, \overline{D}_{2,0} \rangle & (1 < v \leq 2) \\ \langle [\overline{D}_{2,0}, \overline{D}_{3,0}], [\overline{D}_{1,0}, \overline{D}_{3,0}], [\overline{D}_{1,0}, \overline{D}_{2,0}], \overline{D}_{3,0} \rangle & (2 < v \leq 2\frac{1}{5}) \\ \langle [\overline{D}_{2,0}, \overline{D}_{3,0}], [\overline{D}_{1,0}, \overline{D}_{3,0}], \overline{D}_{3,0} + \frac{1}{2}[\overline{D}_{1,0}, \overline{D}_{2,0}] \rangle & (2\frac{1}{5} < v \leq 3) \\ \langle [\overline{D}_{2,0}, \overline{D}_{3,0}], [\overline{D}_{1,0}, \overline{D}_{3,0}] \rangle & (3 < v \leq 4) \\ \langle [\overline{D}_{2,0}, \overline{D}_{3,0}] \rangle & (4 < v \leq 5) \\ \langle 0 \rangle & (5 < v). \end{cases}$$

It follows that the ramification breaks of E/K are $1, 2, 2\frac{1}{5}, 3, 4, 5$.

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